Control System Optimisation: Criteria and Analytical Methods

A common problem in control system design is establishing the appropriate value of controller gain. In general a low value of gain produces a slow system response, while high gain values can cause an excessively-oscillatory response with the possibility of instability. Somewhere between these extremes is a value of gain that produces the best system response. In this Unit we will begin by examining what might be meant by the term 'best response'. An analytical method of determining one particular best response for a simple second-order system is described. Later Units introduce numerical methods for finding the best response in higher-order systems, including systems in which there are major non-linearities.

Best Response

The essential function of a feedback control system is to reduce the error, *e(t)*, between any variable and its demanded value to zero as quickly as possible. Therefore, any criterion used to measure the quality of system response must take into account the variation of *e* over the whole range of time. Four basic criteria are in common use:

Integral of absolute error (IAE) = $\int_0^\infty |e(t)|.dt$ Integral of squared error (ISE) = $\int_0^\infty \{e(t)\}^2.dt$

Integral of time multiplied by absolute error (ITAE) = $\int_0^\infty t|e(t)|.dt$

Integral of time multiplied by squared error (ISE) = $\int_0^\infty t\{e(t)\}^2.dt$

For any of the possible criteria, the best response corresponds to the minimum value of the chosen criterion. Note that in all cases it is either the absolute error or the squared error which is involved: straightforward integration of the error would produce zero result, even if the system response was a constant amplitude oscillation. IAE is often used where digital simulation of a system is being employed, but it is inapplicable for analytical work, because the absolute value of an error function is not generally analytic in form. This problem is overcome by the ISE criterion. The ITAE and ITSE have an additional time multiplier of the error function, which emphasises long-duration errors, and therefore these criteria are most often applied in systems requiring a fast settling time.

Application of the Integral of Squared Error Criterion to Gain Optimisation

Let's consider as an example a simple first order plant with an integral controller, having variable gain *k*, and unity feedback, as illustrated in Fig. 1. If the system is being used as a regulator with zero input, then in the steady-state $x_1 = x_2 = 0$. Because we would like x_1 and x_2 to reduce to zero as quickly as possible, it is appropriate to include both variables in the optimisation process. In the interests of producing analytical results, the Integral of Squared Error Criterion is used:

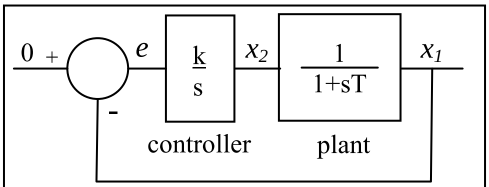

Fig. 1 System for gain optimisation

$$\text{ISE} = \int_0^\infty \{x_1^2 + x_2^2\}.dt \qquad (1)$$

with the two state variables equally weighted. In situations where one state is more significant than the other it is possible to use a weighting factor, *w*, to modify the ISE. For example, ISE= $\int_0^\infty \{wx_1^2 + x_2^2\}.dt$

For the system in Fig. 1 with state variables x_1, x_2 the state space representation is:

$$s\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/T & 1/T \\ -k & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (2)$$

An analytical solution for (1) can be found if it is written in the form:

$$ISE = \int_0^\infty \{ax_1\dot{x}_1 + b(x_1\dot{x}_2 + \dot{x}_1x_2) + cx_2\dot{x}_2\}.dt$$
$$= \left[\tfrac{1}{2}ax_1^2 + bx_1x_2 + \tfrac{1}{2}cx_2^2\right]_0^\infty \quad (3)$$

where a,b,c are constants which can be determined by comparing (1) and (3):

$$ax_1\dot{x}_1 + b(x_1\dot{x}_2 + \dot{x}_1x_2) + cx_2\dot{x}_2 = x_1^2 + x_2^2 \quad (4)$$

but from (2):

$$\dot{x}_1 = (-x_1/T) + (x_2/T) \quad \text{and} \quad \dot{x}_2 = -kx_1$$

and substituting into (4):

$$ax_1\{(-x_1/T) + (x_2/T)\} + b[x_1\{-kx_1\} + x_2\{(-x_1/T) + (x_2/T)\}] + cx_2\{-kx_1\} = x_1^2 + x_2^2$$

$$\therefore -ax_1^2/T + ax_1x_2/T - bkx_1^2 - bx_1x_2/T + bx_2^2/T - ckx_1x_2 = x_1^2 + x_2^2$$

and equating coefficients of x_1^2, x_1x_2, x_2^2 :

$$-a/T - bk = 1 \quad ; \quad a/T - b/T - ck = 0 \quad ; \quad b/T = 1$$

so:

$$a = -T(1 + Tk) \quad ; \quad b = T \quad ; \quad c = -T - (2/k) \quad (5)$$

Substituting the values of a,b,c from (5) into (3):

$$ISE = \left[\tfrac{1}{2}\{-T(1 + Tk)\}x_1^2 + Tx_1x_2 + \tfrac{1}{2}\{-T - (2/k)\}x_2^2\right]_0^\infty \quad (6)$$

The response of the second-order system is always stable, so as $t \to \infty$, $x_1, x_2 \to 0$ and if the initial values of the two states are x_{10}, x_{20}, then from (6):

$$ISE = \tfrac{1}{2}\{T(1 + Tk)\}x_{10}^2 - Tx_{10}x_{20} + \tfrac{1}{2}\{+T + (2/k)\}x_{20}^2 \quad (7)$$

Finally, we can find the appropriate value of gain, k, to minimise the integral of squared error (ISE) for any set of initial conditions and system time constant, T:

$$\frac{d(ISE)}{dk} = \tfrac{1}{2}T^2x_{10}^2 - \{1/k^2\}x_{20}^2 = 0 \text{ at the minimum}$$

So the value of k which minimises the ISE is given by:

$$k^2 = \frac{2}{T^2}\left\{\frac{x_{20}^2}{x_{10}^2}\right\} \quad (8)$$

Numerical example

Consider the numerically simple case of the system in Fig. 1 with a time constant T = 1.0 s, with initial conditions $x_{10} = x_{20} = 1$. System responses for various values of gain, k, are shown in Figs. 2-4.

For low values of gain (k=0.5 in Fig.2) the system response is slow, though there is little overshoot, giving an ISE = 2.26. As the gain increases (k=2.0 in Fig. 3) the response is faster, but there are several overshoots, which contribute to the integral of squared error, so in this case ISE = 1.54. Further gain increases (k = 4.0 in Fig. 4) cause even more overshoot, particularly in x_2, so here the ISE = 2.36. From (8) above the predicted optimum value of gain to minimise the ISE is k = √2. This result can be verified by plotting ISE against k, as shown in Fig. 5, in which the minimum value of ISE around k = 1.414 can be seen. The corresponding response (Fig. 6) is oscillatory but well-damped, so the steady-state is attained quickly.

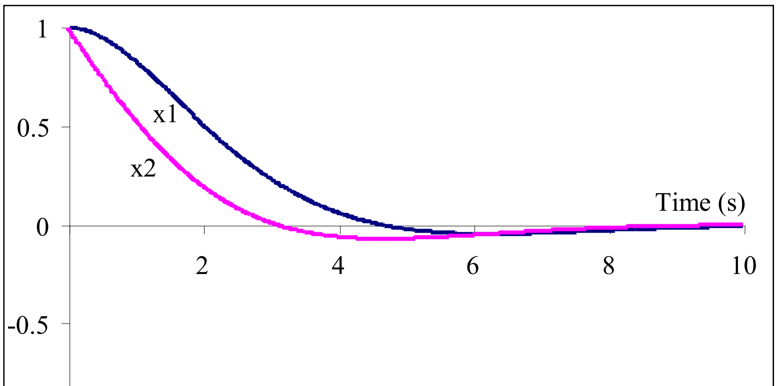

Fig. 2 System responses for k = 0.5

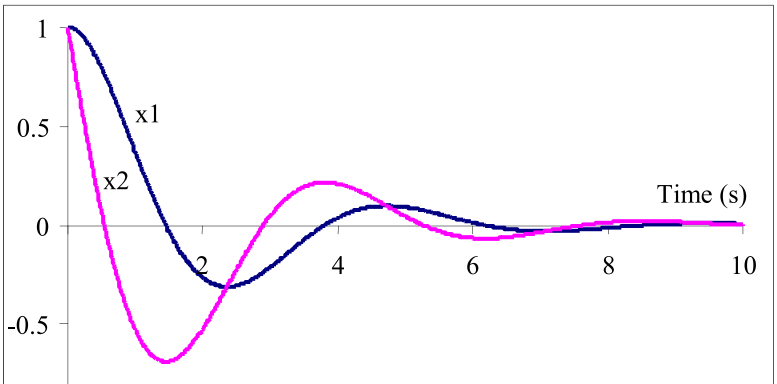

Fig. 3 System responses for k = 2.0

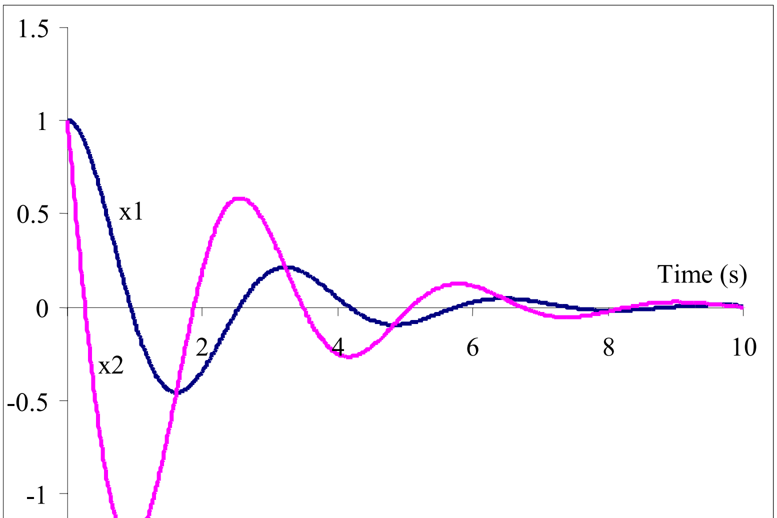

Fig. 4 System responses for k = 4.0

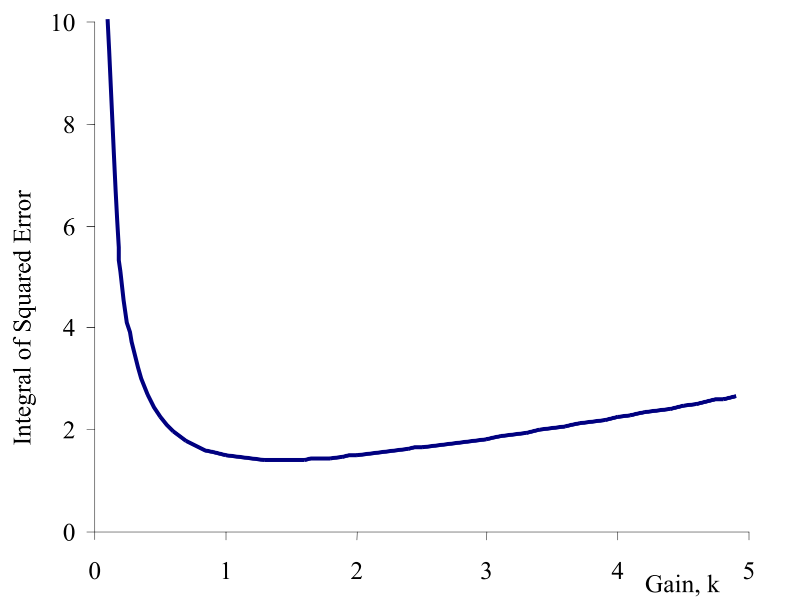

Fig. 5 Integral of Squared Error vs Gain

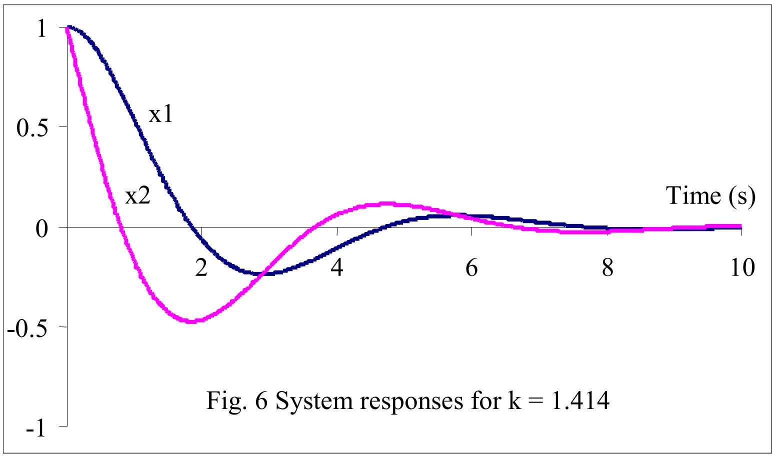

Fig. 6 System responses for k = 1.414